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The box–ball system and the *N*-soliton solution of the ultradiscrete KdV equation

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Abstract

Any state of the box–ball system (BBS) together with its time evolution is described by the *N*-soliton solution (with appropriate choice of *N*) of the ultradiscrete KdV equation. It is shown that simultaneous elimination of all '10'-walls in a state of the BBS corresponds exactly to reducing the parameters that determine 'the size of a soliton' by one. This observation leads to an expression for the solution to the initial-value problem (IVP) for the BBS. Expressions for the solution to the IVP for the ultradiscrete Toda molecule equation and the periodic BBS are also presented.

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1. Motivation and results

A state of the box–ball system (BBS) of Takahashi–Satsuma [1] evolves in time according to an equation which is an ultradiscretization of the discrete KdV equation (called the ultradiscrete KdV equation or UDKdV for short); a class of soliton solutions of the latter describes a state of the BBS together with its time evolution. This was already pointed out in the first paper on the ultradiscretization [2]. Since then there appeared many research papers on the BBS and its extensions [3–20], but it seems that research on the connection with the soliton solution has not been pursued so much. Our motivation was to investigate that connection, and we obtained the following results:

(1) For any given state of the BBS, we show how to identify the parameters of the soliton solution of the UDKdV in order to describe the state and its time evolution.
(2) Simultaneous elimination of all '10'-walls in a state of the BBS corresponds exactly to reducing the parameters that determine 'the size of a soliton' by one.
(3) The above result enables one to determine a 'permutation of solitons'. This leads to expressions for the solution to the initial-value problem (IVP) for the BBS and for the ultradiscrete Toda molecule equation.

(4) Any state of the periodic BBS (PBBS) [9] can be constructed as a restriction of a limit of a sequence of states of the BBS. Through this route, an expression for the solution to the IVP for the PBBS is obtained.

In the following preparatory section, we derive several properties of the N-soliton solution of the UDKdV which will play a role in later sections. In sections 3–6 we state and prove the above-mentioned results.

2. The N-soliton solution of the ultradiscrete KdV equation

The equation

$$\tau_{n+1}^{t+1}\tau_n^{t-1} = (1-\delta)\tau_{n+1}^t\tau_n^t + \delta\tau_{n+1}^{t-1}\tau_n^{t+1} \tag{1}$$

is known as the discrete KdV equation (in 'bilinear form') where n and t are space and time variable, respectively, assumed usually to be integers and τ is real valued. Applying a limiting procedure known as the ultradiscretization [2] we have

$$\rho_{n+1}^{t+1} + \rho_n^{t-1} = \max\left\{\rho_{n+1}^t + \rho_n^t, \rho_{n+1}^{t-1} + \rho_n^{t+1} - 1\right\}, \tag{2}$$

where we have put $\delta = \exp(-1/\epsilon)$ and assumed the existence of $\rho_n^t = \lim_{\epsilon\to+0}\epsilon\log\tau_n^t$. We call this the ultradiscrete KdV equation (UDKdV). The N-soliton solution of (1) is known, for example, by virtue of a general theory described in [21]. Ultradiscretizing that solution yields a solution of the UDKdV, which we call the N-soliton solution of the UDKdV, having the following form [2]: for positive integer N and real parameters $P_1,\ldots,P_N$ and $\theta_1,\ldots,\theta_N$,

$$\begin{aligned}\eta_n^t &= \eta_n^t\left((P_i,\theta_i)_{i=1}^N\right) = \eta_n^t(N; P_1,\ldots,P_N;\theta_1,\ldots,\theta_N)\\ &= \max\left\{0, \max_{\substack{J\subset[N]\\ J\neq\emptyset}}\left[\sum_{i\in J}(\theta_i + tP_i - n) - \sum_{\substack{i,j\in J\\ i\neq j}}\min\{P_i,P_j\}\right]\right\},\end{aligned} \tag{3}$$

where $[N] = \{1,2,\ldots,N\}$ and $\max_P[\cdots]$ stands for $\max\{\cdots|P\}$. If all P_i's and θ_i's are integer then so is η_n^t, but we will not assume so unless otherwise stated. Also we will not restrict n and t to be integer. In this section, we explore the properties of this function $\mathbb{R}\to\mathbb{R}, n\mapsto\eta_n^t$.

First, it is convenient to rewrite (3). Let $k = 0,1,\ldots,N$ and let J be a k-element subset (k-subset) of $[N]$. Define $\Phi_k^t(J) = \Phi_k^t\left(J;(P_i,\theta_i)_{i=1}^N\right)$ to be

$$\Phi_k^t(J) = \sum_{i\in J}(\theta_i + tP_i) - \sum_{\substack{i,j\in J\\ i\neq j}}\min\{P_i,P_j\} \qquad (k\geqslant 1) \tag{4}$$

and $\Phi_0^t(\emptyset) = 0$, and define $\Psi_k^t = \Psi_k^t\left((P_i,\theta_i)_{i=1}^N\right)$ to be

$$\Psi_k^t = \max_{\substack{J\subset[N]\\ |J|=k}}\left[\Phi_k^t(J)\right] \qquad (k\geqslant 1) \tag{5}$$

and $\Psi_0^t = 0$. Then

$$\eta_n^t = \max\left\{\Psi_k^t - kn \,\middle|\, k = 0,1,\ldots,N\right\}. \tag{6}$$

The following proposition is deduced immediately.

Proposition 2.1.

(a) The function $n\mapsto\eta_n^t$ is convex and piecewise linear (namely its graph is a polygonal curve with finitely many vertices).

(b) There exist n_1 *and* n_2 *such that* $\eta_n^t = 0$ *for* $n > n_1$*, and* $\eta_n^t = \Psi_N^t - Nn$ *for* $n < n_2$.
(c)

$$\eta_n^t(N; \underbrace{0, \ldots, 0}_{N}; \theta_1, \ldots, \theta_N) = \max\left\{0, \max_{\emptyset \neq J \subset [N]} \left[\sum_{i \in J} (\theta_i - n)\right]\right\}.$$

Note that this does not depend on t.
(d) For $N > N' > 0$

$$\eta_n^t(N; P_1, \ldots, P_{N'}, \underbrace{0, \ldots, 0}_{N-N'}; \theta_1, \ldots, \theta_N)$$
$$= \eta_n^t(N'; P_1, \ldots, P_{N'}; \theta_1, \ldots, \theta_{N'}) + \eta_n^t(N - N'; \underbrace{0, \ldots, 0}_{N-N'}; \theta_{N'+1}, \ldots, \theta_N).$$

Note that, due to (c)*, the second term on the right-hand side does not depend on t.*

Proof. (a) and (b) are clear from (6). (c) follows from (3). Proof of (d): write $(N', N]$ for $\{i | i \in \mathbb{Z}, N' < i \leqslant N\}$. Let $\xi_i = \theta_i + tP_i - n$ if $i \in [N']$, $\xi_i = \theta_i - n$ if $i \in (N', N]$. Let $\xi_{ij} = \min\{P_i, P_j\}$. There follows

$$\max\left\{0, \max_{\emptyset \neq J \subset [N]}\left[\sum_{i \in J \cap [N']} \xi_i + \sum_{i \in J \cap (N', N]} \xi_i - \sum_{\substack{i, j \in J \cap [N'],\\ i \neq j}} \xi_{ij}\right]\right\}$$
$$= \max\left\{0, \max_{\emptyset \neq J_1 \subset [N']}\left[\sum_{i \in J_1} \xi_i - \sum_{\substack{i, j \in J_1,\\ i \neq j}} \xi_{ij}\right], \max_{\emptyset \neq J_2 \subset (N', N]}\left[\sum_{i \in J_2} \xi_i\right],\right.$$
$$\left.\max_{\substack{\emptyset \neq J_1 \subset [N']\\ \emptyset \neq J_2 \subset (N', N]}}\left[\sum_{i \in J_1} \xi_i - \sum_{\substack{i, j \in J_1,\\ i \neq j}} \xi_{ij} + \sum_{i \in J_2} \xi_i\right]\right\}$$
$$= \max\left\{0, \max_{\emptyset \neq J_1 \subset [N']}\left[\sum_{i \in J_1} \xi_i - \sum_{\substack{i, j \in J_1,\\ i \neq j}} \xi_{ij}\right], \max_{\emptyset \neq J_2 \subset (N', N]}\left[\sum_{i \in J_2} \xi_i\right],\right.$$
$$\left.\max_{\emptyset \neq J_1 \subset [N']}\left[\sum_{i \in J_1} \xi_i - \sum_{\substack{i, j \in J_1,\\ i \neq j}} \xi_{ij}\right] + \max_{\emptyset \neq J_2 \subset (N', N]}\left[\sum_{i \in J_2} \xi_i\right]\right\}$$
$$= \max\left\{0, \max_{\emptyset \neq J_1 \subset [N']}\left[\sum_{i \in J_1} \xi_i - \sum_{\substack{i, j \in J_1,\\ i \neq j}} \xi_{ij}\right]\right\} + \max\left\{0, \max_{\emptyset \neq J_2 \subset (N', N]}\left[\sum_{i \in J_2} \xi_i\right]\right\},$$

where we have used

$$\{J | \emptyset \neq J \subset [N]\} = \{J \subset [N] \big| J \cap [N'] \neq \emptyset, J \cap (N', N] = \emptyset\}$$
$$\bigcup \{J \subset [N] | J \cap [N'] = \emptyset, J \cap (N', N] \neq \emptyset\}$$
$$\bigcup \{J \subset [N] | J \cap [N'] \neq \emptyset, J \cap (N', N] \neq \emptyset\}$$

and $\max\{0, A, B, A + B\} = \max\{0, A\} + \max\{0, B\}$. □

Define $A_k^t = A_k^t\left((P_i, \theta_i)_{i=1}^N\right) = A_k^t(N; P_1, \ldots, P_N; \theta_1, \ldots, \theta_N)$ as

$$A_k^t = \Psi_k^t - \Psi_{k-1}^t \qquad (k = 1, 2, \ldots, N), \tag{7}$$

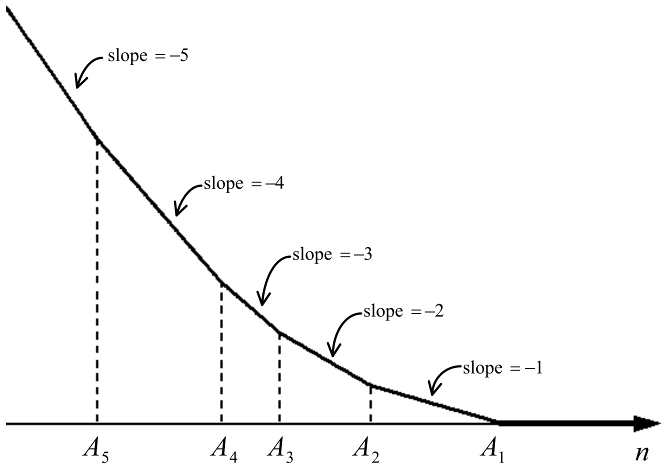

Figure 1. The function $n \mapsto \eta_n^t$, (10).

so that $\Psi_k^t = \sum_{i=1}^k A_i^t$. Define $A_0^t = +\infty$, $A_{N+1}^t = -\infty$ for convenience.

Proposition 2.2. *If $P_i \geqslant 0\,(i = 1, \ldots, N)$ then*

$$A_1^t \geqslant A_2^t \geqslant \cdots \geqslant A_N^t; \tag{8}$$

if the stronger condition $P_i > 0\,(i = 1, \ldots, N)$ holds then

$$A_1^t > A_2^t > \cdots > A_N^t. \tag{9}$$

In either case the function (6), i.e., (3), has a linear form

$$\eta_n^t = \Psi_k^t - kn \tag{10}$$

in each interval $n \in \left[A_{k+1}^t, A_k^t\right] = \left\{n \in \mathbb{R} \,\middle|\, A_{k+1}^t \leqslant n \leqslant A_k^t\right\}$ where $k = 0, 1, \ldots, N$.

A sketch of the graph of (10) is shown in figure 1. For a proof, we introduce the concept of 'permutations of solitons', which will play a crucial role in later sections. For each $k \in [N]$ we regard $J \mapsto \Phi_k^t(J)$ as a function on the set of all k-subsets of $[N]$.

Lemma 2.3. *Let $k = 1, 2, \ldots, N-1$. Let $\{c_1, \ldots, c_k\}$ be a k-subset of $[N]$ where the function $J \mapsto \Phi_k^t(J)$, (4), attains its maximum. Then there exists a number $c_{k+1} \in [N]$, $c_{k+1} \notin \{c_1, \ldots, c_k\}$, such that $J \mapsto \Phi_{k+1}^t(J)$ attains its maximum at $\{c_1, \ldots, c_{k+1}\}$.*

Proof. Let $\{d_1, \ldots, d_{k+1}\}$ be a $(k+1)$-subset of $[N]$ where Φ_{k+1}^t attains its maximum. Let the number of elements of $\{d_1, \ldots, d_{k+1}\} \cap \{c_1, \ldots, c_k\}$ be r. By renaming, if necessary, we can assume that (a) $d_i \in \{c_1, \ldots, c_k\}$ if $i \leqslant r$, (b) $d_i \notin \{c_1, \ldots, c_k\}$ if $r < i \leqslant k+1$, and (c) $P_{d_{k+1}} = \min\{P_{d_i} | r < i \leqslant k+1\}$. We shall show

$$\Phi_{k+1}^t(\{d_1, \ldots, d_{k+1}\}) = \Phi_{k+1}^t(\{c_1, \ldots, c_k, d_{k+1}\}). \tag{11}$$

From trivial inequalities, $\min\left\{P_j, P_{d_{k+1}}\right\} \leqslant P_{d_{k+1}}$ $(\forall\, j)$ and results of (c), $P_{d_{k+1}} = \min\left\{P_{d_i}, P_{d_{k+1}}\right\}$ $(r < i \leqslant k)$, we have

$$\sum_{i=r+1}^{k} \min\left\{P_{d_i}, P_{d_{k+1}}\right\} \geqslant \sum_{j \in \{c_1,\ldots,c_k\} \setminus \{d_1,\ldots,d_r\}} \min\left\{P_j, P_{d_{k+1}}\right\}.$$

Adding $\sum_{i=1}^{r} \min\{P_{d_i}, P_{d_{k+1}}\}$ to both sides of the inequality yields $\sum_{i=1}^{k} \min\{P_{d_i}, P_{d_{k+1}}\} \geqslant \sum_{i=1}^{k} \min\{P_{c_i}, P_{d_{k+1}}\}$, and hence

$$\left(\theta_{d_{k+1}} + tP_{d_{k+1}}\right) - \sum_{i=1}^{k} 2\min\{P_{d_i}, P_{d_{k+1}}\} \leqslant \left(\theta_{d_{k+1}} + tP_{d_{k+1}}\right) - \sum_{i=1}^{k} 2\min\{P_{c_i}, P_{d_{k+1}}\}.$$

$\Phi_k^t(\{d_1, \ldots, d_k\}) \leqslant \Phi_k^t(\{c_1, \ldots, c_k\})$ by the assumption on c_i. From those two inequalities we have $\Phi_{k+1}^t(\{d_1, \ldots, d_{k+1}\}) \leqslant \Phi_{k+1}^t(\{c_1, \ldots, c_k, d_{k+1}\})$. The opposite inequality also holds because of the assumption on d_i. Thus we have established (11) and the lemma follows. □

Definition 2.4. *Let c_1 be a number where the function $c \mapsto \Phi_1^t(\{c\}) = \theta_c + tP_c$ on $[N]$ attains its maximum; choose any one of such numbers if there are many such ones. For each $k = 2, \ldots, N$, choose a number $c_k \in [N], c_k \notin \{c_1, \ldots, c_{k-1}\}$, recursively such that Φ_k^t attains its maximum at $\{c_1, \ldots, c_k\}$ (again it is not unique in general); the existence of such c_k is assured by the preceding lemma. Then $i \mapsto c_i$ defines a permutation of $[N]$. The set of all such permutations is denoted by*

$$\mathfrak{S}_N^t\left((P_i, \theta_i)_{i=1}^N\right) \qquad (12)$$

or $\mathfrak{S}_N^t$ for short.

Lemma 2.5. *Let $\sigma_t \in \mathfrak{S}_N^t$. Then*

(a) $\Psi_k^t = \Phi_k^t(\{\sigma_t(1), \ldots, \sigma_t(k)\}) = \sum_{i=1,\ldots,k} \left(\theta_{\sigma_t(i)} + tP_{\sigma_t(i)}\right) - \sum_{\substack{i,j=1,\ldots,k \\ i \neq j}} \min\{P_{\sigma_t(i)}, P_{\sigma_t(j)}\}$;

(b) $A_k^t = \theta_{\sigma_t(k)} + tP_{\sigma_t(k)} - \sum_{i=1}^{k-1} 2\min\{P_{\sigma_t(i)}, P_{\sigma_t(k)}\}$;

(c) if $k < l$ then $A_k^t \geqslant \theta_{\sigma_t(l)} + tP_{\sigma_t(l)} - \sum_{i=1}^{k-1} 2\min\{P_{\sigma_t(i)}, P_{\sigma_t(l)}\}$.

Proof. (a) and (b) are apparent. By the construction of σ_t we have

$$\Phi_k^t(\{\sigma_t(1), \ldots, \sigma_t(k)\}) \geqslant \Phi_k^t(\{\sigma_t(1), \ldots, \sigma_t(k-1), \sigma_t(l)\}).$$

Subtracting $\Phi_{k-1}^t(\{\sigma_t(1), \ldots, \sigma_t(k-1)\})$ from both sides yields the inequality of (c). □

Proof of proposition 2.2. Let $P_i \geqslant 0\ (\forall\, i)$. Then $\min\{P_{\sigma_t(k)}, P_{\sigma_t(k+1)}\} \geqslant 0$, so that

$$\theta_{\sigma_t(k+1)} + tP_{\sigma_t(k+1)} - \sum_{i=1}^{k-1} 2\min\{P_{\sigma_t(i)}, P_{\sigma_t(k+1)}\}$$
$$\geqslant \theta_{\sigma_t(k+1)} + tP_{\sigma_t(k+1)} - \sum_{i=1}^{k} 2\min\{P_{\sigma_t(i)}, P_{\sigma_t(k+1)}\} = A_{k+1}^t.$$

The left-hand side is not larger than A_k^t by lemma 2.5 (c) and hence (8) follows. In a similar manner we obtain (9) when $P_i > 0\ (\forall\, i)$.

Now assume inequalities (8), $A_1^t \geqslant A_2^t \geqslant \cdots \geqslant A_N^t$. We shall write A_i for A_i^t because the dependence on t is irrelevant. Let l_k denote a straight line in the nY-plane defined by $Y = f_k(n)$ where $f_k(n) = \Psi_k - kn = \sum_{i=1}^{k} A_i - kn\ (k = 0, 1, \ldots, N)$. We want to show that

$$\max\{f_i(n) \mid i = 0, 1, \ldots, N\} = f_k(n) \qquad \text{for} \quad n \in [A_{k+1}, A_k]$$

holds for every $k = 0, 1, \ldots, N$. To this end, it is sufficient to show: (i) $f_k \geqslant f_h$ holds on $[A_{k+1}, \infty)$ for every pair of $k \in \{0, 1, \ldots, N-1\}$ and $h \in \{0, 1, \ldots, N\}$ such that $h \geqslant k$; and (ii) $f_k \geqslant f_h$ holds on $(-\infty, A_k]$ for every pair of $k \in \{1, \ldots, N\}$ and $h \in \{0, 1, \ldots, N\}$ such that $h \leqslant k$.

The proof of (i) goes as follows: let $\in \{0, 1, \ldots, N-1\}$. Two lines l_{k+1} and l_k intersect at the point with coordinates $(A_{k+1}, f_k(A_{k+1}))$. Let $h \geqslant k$; the line l_h crosses the horizontal line $Y = f_k(A_{k+1})$ at the point with n-coordinate $\frac{1}{h}\left(\sum_{k<i\leqslant h} A_i + kA_{k+1}\right)$; since this coordinate is not larger than A_{k+1} (and the slope of l_h is not larger than that of l_k), we have the assertion of (i). Since l_k and l_{k-1} intersect at $n = A_k$, we have $f_k \geqslant f_{k-1}$ on $(-\infty, A_k]$. This leads to the assertion of (ii). □

The following two propositions will be referred to later.

Proposition 2.6. *Assume that the parameters satisfy the conditions*

$$P_1 \geqslant P_2 \geqslant \cdots \geqslant P_N > 0$$

and

$$P_i = P_j, i < j \Longrightarrow \theta_i \geqslant \theta_j.$$

Let

$$\Lambda = \Lambda\left((P_i)_{i=1}^N\right) = \{(i,k) | i \in [N], k \in [N], i < k, \text{ and } P_i > P_k\}.$$

If t satisfies

$$t \geqslant \frac{\theta_k - \theta_i}{P_i - P_k} + 2(i-1) \qquad \text{for all} \quad (i,k) \in \Lambda \tag{13}$$

then the set $\mathfrak{S}_N^t\left((P_i, \theta_i)_{i=1}^N\right)$ contains the identity $i \mapsto i$, so that we have

$$A_k^t = \theta_k + tP_k - 2(k-1)P_k \qquad (k = 1, \ldots, N). \tag{14}$$

Proof. Writing explicitly the condition for σ_t to be $\sigma_t(i) = i$ for each i condition (13) is obtained: namely, we can choose the value $\sigma_t(1)$ as 1 if $\theta_1 + tP_1 \geqslant \theta_k + tP_k$ for $k > 1$, that is,

$$t \geqslant \frac{\theta_k - \theta_1}{P_1 - P_k} \quad \text{for all } k \quad \text{satisfying} \quad P_1 > P_k \quad \text{and} \quad k > 1.$$

When the above condition holds we can choose $\sigma_t(2)$ as 2 if $\theta_2 + tP_2 - 2P_2 \geqslant \theta_k + tP_k - 2P_k$ for $k > 1$, that is,

$$t \geqslant \frac{\theta_k - \theta_2}{P_2 - P_k} + 2 \quad \text{for all } k \quad \text{satisfying} \quad P_2 > P_k \quad \text{and} \quad k > 2.$$

Continuing the argument condition (13) follows. □

Proposition 2.7. $\mathfrak{S}_N^t\left((P_i, \theta_i)_{i=1}^N\right) = \mathfrak{S}_N^t\left((P_i + x, \theta_i)_{i=1}^N\right)$ *for any $x \in \mathbb{R}$.*

Proof. Recall (4). For any k and $J \subset [N], |J| = k$, we have

$$\Phi_k^t\left(J; (P_i + x, \theta_i)_{i=1}^N\right) = \Phi_k^t\left(J; (P_i, \theta_i)_{i=1}^N\right) + xk(t - k + 1).$$

Thus if J maximizes $\Phi_k^t\left(J; (P_i, \theta_i)_{i=1}^N\right)$ then it maximizes $\Phi_k^t\left(J; (P_i + x, \theta_i)_{i=1}^N\right)$ as well. The converse is also true. Hence the proposition holds. □

For each i the equation $n = A_i^t$ defines a polygonal curve in the nt-plane; an example of such curves is shown in figure 2.

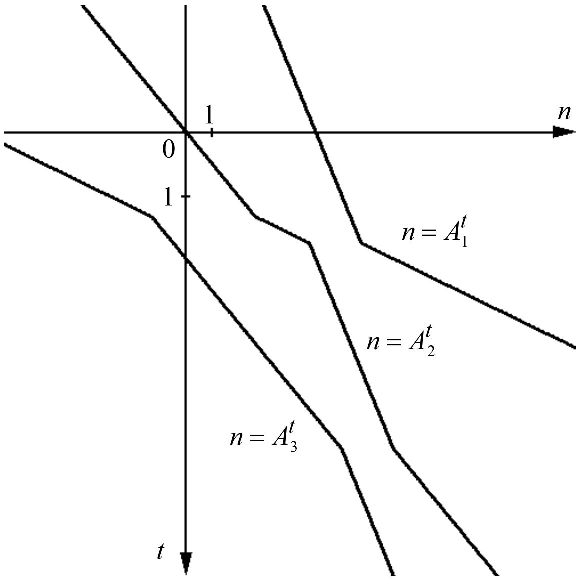

Figure 2. Polygonal curves in the nt-plane defined by the equations $n = A_i^t, i = 1, 2, 3$, where $(P_1, P_2, P_3, \theta_1, \theta_2, \theta_3) = (5, 2, 1, -2, 2, 5)$.

3. Dynamics of the box–ball system and soliton solutions of the ultradiscrete KdV equation

The box–ball system (BBS) of Takahashi–Satsuma [1] is a discrete dynamical system (Ω, T). The set of states Ω is defined by

$$\Omega = \{f \mid f : \mathbb{Z} \to \{0, 1\} \text{ where the set } f^{-1}(\{1\}) \text{ is finite}\}.$$

We often represent $f \in \Omega$ as a sequence of 0 and 1, and write

$$\cdots f(-2) f(-1) f(0) f(1) f(2) \cdots.$$

The mapping $T : \Omega \to \Omega$, called the time evolution operator, is defined as explained in figure 3. T is invertible, as is easily seen from our 'pictorial' definition. It is known that the following is equivalent to the definition of T: for $f \in \Omega$, let n_0 be a lower bound of $f^{-1}(\{1\})$; then $Tf \in \Omega$ is determined by the conditions: $(Tf)(n) = 0$ for $n < n_0$, and

$$(Tf)(n) = \min\left\{1 - f(n), \sum_{k=-\infty}^{n-1} f(k) - \sum_{k=-\infty}^{n-1} (Tf)(k)\right\} \qquad \text{for} \quad n \geqslant n_0. \tag{15}$$

Thinking of $f \in \Omega$ as a state at $t = 0$ we call $T^t f = T^t(f)$ the state at t $(t \in \mathbb{Z})$ where T^t is t times composition of T if $t > 0$, or $|t|$ times composition of T^{-1} if $t < 0$. Sometimes we write f^t for $T^t(f)$.

Let $f \in \Omega$. Let $i, m \in \mathbb{Z}$ and $m > 0$. We say f has a block (of 1's) of length m at position i if $f(j) = 1$ for $i - m \leqslant j \leqslant i - 1$, and $f(i) = f(i - m - 1) = 0$. (The term 'block' is used in a different meaning from that in [10].) We say f has (or there is) a '10'-wall at position i if $f(i - 1) = 1$ and $f(i) = 0$. The number of blocks contained in f is denoted by $p_1(f)$. Positions of blocks are denoted by $a_1(f) > a_2(f) > \cdots > a_{p_1(f)}(f)$. The block at position $a_i(f)$ is called the ith block. It is convenient to consider that p_1 is a mapping of Ω into $\mathbb{Z}_{\geqslant 0}$.

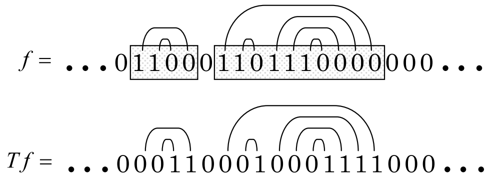

Figure 3. Definition of T: let $f \in \Omega$. (a) In the sequence f find a pair of positions i and $i+1$ such that $f(i) = 1$ and $f(i+1) = 0$, and mark them; repeat the same procedure until all such pairs are marked. (b) Skipping the marked positions we get a subsequence of f; for this subsequence repeat the same process of marking positions, so that we get another marked subsequence. (c) Repeat (b) until one obtains a subsequence consisting only of 0's. The situation is depicted in the upper figure. After these preparatory processes, change all values at the marked positions simultaneously; then we get a sequence Tf. (The term 'marked position' will be used again in section 6.)

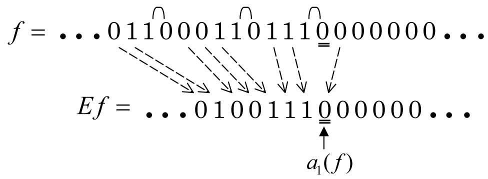

Figure 4. The '10'-elimination.

Given f, a state $Ef = E(f)$ is defined to be

$$(Ef)(n) = \begin{cases} f(n+1) & (n \geqslant a_1(f)), \\ f(n-2k+1) & (a_{k+1}(f) + 2k \leqslant n \leqslant a_k(f) + 2k - 3; \\ & \quad k = 1, 2, \ldots, p_1(f) - 1), \\ f(n - 2p_1(f) + 1) & (n \leqslant a_{p_1(f)} + 2p_1(f) - 3). \end{cases}$$

The mapping $E : \Omega \to \Omega$ is called the '10'-elimination. Ef is a subsequence of f obtained by eliminating all '10'-walls in f simultaneously (see figure 4). Define $p_i(f), i = 2, 3, \ldots,$ to be

$$p_i(f) = p_1(E^{i-1}(f)).$$

As is easily seen $p_i(T(f)) = p_i(f)$ hold, so that they are conserved quantities of the BBS. Since $\sum_{i=1}^{\infty} p_i(f) = |f^{-1}(\{1\})|$ and $p_1(f) \geqslant p_2(f) \geqslant p_3(f) \geqslant \cdots$, the sequence $p(f) = (p_1(f), p_2(f), \ldots)$ is a partition of the integer $|f^{-1}(\{1\})|$. The conjugate of the partition $p(f)$ is denoted by $L(f) = (L_1(f), L_2(f), \ldots)$, that is, $L_j(f) = |\{i | i \geqslant 1, p_i(f) \geqslant j\}|$.

The following fact is theorem 1 in [4] with minor modifications, describing states at sufficiently large t.

Fact 3.1. *For $f \in \Omega$, there exists a real number t_0 such that if $t \geqslant t_0$ then*

$$a_i(f^t) - a_k(f^t) - 2(k-i)L_k(f) \geqslant 0 \qquad \textit{for all } i, k \textit{ such that } i < k$$

(so that in particular $a_1(f^t) > a_2(f^t) > \cdots > a_{p_1(f)}(f^t)$), $a_i(f^t) = L_i(f)t + const_i$, and

$$f^t(n) = \begin{cases} 1 & \textit{if} \quad n \in \bigcup_{i=1,\ldots,p_1(f)} [a_i(f^t) - L_i(f), a_i(f^t) - 1], \\ 0 & \textit{otherwise}\ . \end{cases}$$

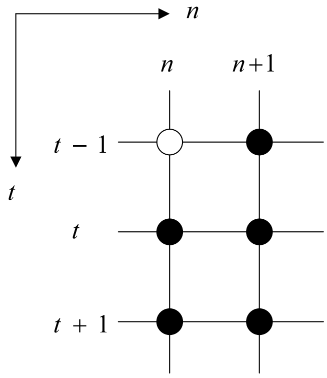

Figure 5. Structure of the UDKdV (2): once the values at the sites of black dots are given then the value at the site of white dot is determined.

Let $f \in \Omega$ and $u_n^t = (T^t f)(n)$. It follows from (15) that $\left(u_n^t\right)$ satisfies

$$u_n^t = \min\left\{1 - u_n^{t-1}, \sum_{k=-\infty}^{n-1} u_k^{t-1} - \sum_{k=-\infty}^{n-1} u_k^t\right\}. \tag{16}$$

This equation is also called the ultradiscrete KdV equation because it is obtained as an ultradiscretization of the discrete KdV equation [2]. Transforming the dependent variable as

$$\rho_n^t = \rho_n^t(u) = \sum_{n'=n}^{\infty} \sum_{t'=-\infty}^{t} u_{n'}^{t'} \tag{17}$$

(the summation is actually finite by definition of Ω and T) ρ_n^t then satisfies (2). We can solve (17) for u as

$$u_n^t = \rho_{n+1}^{t-1} - \rho_{n+1}^t - \rho_n^{t-1} + \rho_n^t. \tag{18}$$

Using $a_i(f^t)$, the position of block in f^t, ρ_n^t can be represented explicitly as

$$\rho_n^t = \begin{cases} 0 & (n \geqslant a_1(f^t)), \\ \sum_{i=0}^{k} a_i(f^t) - kn & (a_{k+1}(f^t) \leqslant n \leqslant a_k(f^t); k = 1, 2, \ldots, p_1(f) - 1), \\ \sum_{i=0}^{p_1(f)} a_i(f^t) - p_1(f)n & (n \leqslant a_{p_1(f)}(f^t)). \end{cases}$$

A close similarity to (10) is apparent and we have the following theorem.

Theorem 3.2. *Let $f \in \Omega$ and let t_0 be as in fact 3.1. Choose the parameters in the soliton solution $\eta_n^t = \eta_n^t\left((P_i, \theta_i)_{i=1}^N\right)$, (3), as $N = p_1(f)$ and*

$$P_i = L_i(f), \qquad \theta_i = a_i(f^{t_0}) - t_0 L_i(f) + 2(i-1)L_i(f) \quad (i = 1, 2, \ldots, p_1(f)). \tag{19}$$

Then η_n^t agrees with $\rho_n^t(u)$, (17), and hence

$$u_n^t = (T^t f)(n) = \eta_{n+1}^{t-1} - \eta_{n+1}^t - \eta_n^{t-1} + \eta_n^t. \tag{20}$$

Proof. ρ_n^t and η_n^t satisfy the same equation (2), which says once the values at the sites of black dots (see figure 5) are given then the value at the site of white dot is determined; thus it is sufficient to show: (a) they agree for all $t > t_0$ and (b) there exists n_0 such that they agree for all $n > n_0$ if $t \leqslant t_0$.

Both ρ_n^t and η_n^t are represented as polygonal curves with N turning points. If $t \geqslant t_0$ the graph of $n \mapsto \rho_n^t$ turns at $a_k(f^t)$ given in fact 3.1, so that $a_k(f^t) = a_k(f^{t_0}) + (t - t_0)L_k(f)$. On the other hand, since $t_0 \geqslant \max\{(\theta_k - \theta_i)/(P_i - P_k) + 2(i-1) | (i, k) \in \Lambda\}$ with Λ as defined in proposition 2.6, if $t \geqslant t_0$ the graph of $n \mapsto \eta_n^t$ turns at A_i^t, (14). Substituting (19) into this expression, $A_k^t = a_k(f^{t_0}) + (t - t_0)L_k(f)$. Thus if $t \geqslant t_0$ then $a_k(f^t) = A_k^t$, hence $\rho_n^t = \eta_n^t$. (a) is proved.

Let $n_0 = a_1(f^{t_0})$; then $\rho_n^{t_0} = \eta_n^{t_0} = 0$ if $n \geqslant n_0$. Hence (since each block moves leftwards by T^{-1}) $\rho_n^t = \eta_n^t = 0$ hold for $n \geqslant n_0$ and $t \leqslant t_0$, which is (b). □

Corollary 3.3. *Choose the parameters as in theorem 3.2. Then the position of the* kth *block is given by*

$$a_k(f^t) = A_k^t = \theta_{\sigma_t(k)} + t P_{\sigma_t(k)} - \sum_{i=1}^{k-1} 2 \min\{P_{\sigma_t(i)}, P_{\sigma_t(k)}\} \qquad (k = 1, \ldots, p_1(f)), \tag{21}$$

where $\sigma_t \in \mathfrak{S}^t_{p_1(f)}\big((P_i, \theta_i)_{i=1}^{p_1(f)}\big)$.

We have obtained an expression for the state of the BBS at arbitrary time t, (20), in which parameters are given by (19), expressed by data at some large enough time t_0.

4. The '10'-elimination and soliton solutions

Let $f \in \Omega$, and consider that it is a state at $t = 0$ of the BBS. Suppose that we have an expression for $T^t f$, the state at arbitrary time t corresponding to the initial state f, as in theorem 3.2 in terms of $p_1(f)$-soliton solutions with parameters $P_1, \ldots, P_{p_1(f)}$ and $\theta_1, \ldots, \theta_{p_1(f)}$. (We should note that $P_i \geqslant 2$ if $i \leqslant p_2(f)$, and $P_i = 1$ if $i > p_2(f)$.) Then $T^t Ef$, the state at time t corresponding to the initial state Ef (which is obtained from f by applying the '10'-elimination), is expressed exactly as in the theorem by $p_2(f)$-soliton solutions with parameters $P_1, \ldots, P_{p_2(f)}$ and $\theta_1, \ldots, \theta_{p_2(f)}$.

Before proving this statement, we note that the two operators T (time evolution) and E ('10'-elimination) are almost commutative.

Lemma 4.1.

$$(ETf)(n+1) = (TEf)(n) \qquad \textit{for} \quad f \in \Omega \quad \textit{and} \quad n \in \mathbb{Z};$$

that is, if we define a shift $S : \Omega \to \Omega$ *as* $(Sf)(n) = f(n+1)$ *then*

$$SET = TE.$$

(Note that clearly we have $ST = TS$ *and* $SE = ES$*.)*

Proof. It is sufficient to show for f such that $Ef \neq 0$. Look at figure 6: compare TEf and a state obtained from Tf by eliminating all '01's (instead of '10's); clearly they agree up to a shift of the space coordinates. On the other hand, ETf and the state obtained from Tf by eliminating '01's also agree up to a shift, for both the '10'-elimination and '01'-elimination reduce the sizes of all blocks of 1's and 0's by 1. Therefore, TEf and ETf agree up to a shift, that is, there exists $\delta(f) \in \mathbb{Z}$ such that $(ETf)(n) = (TEf)(n + \delta(f))$ for all $n \in \mathbb{Z}$.

It remains to be shown that in fact we have $\delta(f) = -1$ for any f. To this end we examine how the position of the first block of 1's moves. We write $Q_1(f)$ for the size of the first block of 1's. If $a_1(Tf) - a_1(f) \geqslant 2$ then both $a_1(TEf) = a_1(Tf) - 1$ and $a_1(ETf) = a_1(Tf)$ hold (the latter is due to $Q_1(Tf) \geqslant 2$), and hence $\delta(f) = a_1(TEf) - a_1(ETf) = -1$ follows. Now consider the case of $a_1(Tf) - a_1(f) = 1$.

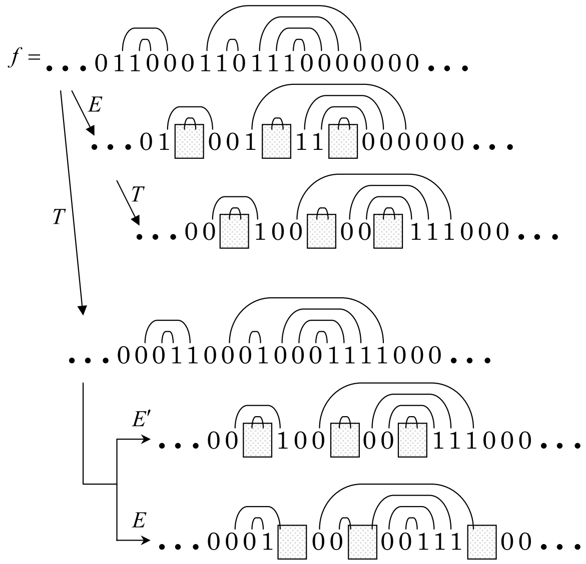

Figure 6. TEf and ETf where E' eliminates '01's.

Let Rf denote the state obtained from f by replacing the rightmost 1 by 0. We note that $a_1(T(R^m f)) - a_1(R^m f) \geqslant 2$ for some m, since if not so we have $Ef = 0$, a contradiction. It is easily seen that $a_1(TEf) = a_1(TERf) + 2$ and $a_1(ETf) = a_1(ETRf) + 2$ hold. Hence $\delta(f) = a_1(TEf) - a_1(ETf) = a_1(TERf) - a_1(ETRf) = \delta(Rf)$ follows, and therefore $\delta(f) = \delta(R^m f) = -1$. □

Theorem 4.2. *Let $f \in \Omega$. Let the parameters P_i, θ_i are chosen as in theorem 3.2. Then the $p_2(f)$-soliton solution $\eta_n^{(1)t} = \eta_n^t\big((P_i - 1, \theta_i)_{i=1}^{p_2(f)}\big)$ agrees with $\rho_n^t(u^{(1)})$, (17), where $u_n^{(1)t} = (T^t(Ef))(n)$, and hence*

$$u_n^{(1)t} = (T^t Ef)(n) = \eta_{n+1}^{(1)t-1} - \eta_{n+1}^{(1)t} - \eta_n^{(1)t-1} + \eta_n^{(1)t}.$$

Proof.

(a) Both $\eta_n^{(1)t}$ and $\rho_n^t(u^{(1)})$ satisfy the same equation (2). It is sufficient to show that they agree for sufficiently large t (because of an analogous reason as in the proof of theorem 3.2). We know that each of them is, for fixed t, a convex piecewise linear function of n with p_2 'turning points' such that the slope increases by 1 when passing through each turning point, and that it becomes the zero function on a right half infinite interval. Therefore we need to prove that the coordinate of the kth turning point of the former, $A_k^t\big((P_i - 1, \theta_i)_{i=1}^{p_2(f)}\big)$, coincides with that of the latter, $a_k(T^t Ef)$, for each $k = 1, \ldots, p_2(f)$.

(b) It follows that

$$A_k^t\big((P_i - 1, \theta_i)_{i=1}^{p_1(f)}\big) = A_k^t\big((P_i, \theta_i)_{i=1}^{p_1(f)}\big) - t + 2(k-1) \quad \text{for} \quad k = 1, \ldots, p_1(f) \qquad (22)$$

from lemma 2.5 (b) and proposition 2.7.

Proposition 2.1 (d) implies

$$\eta_n^t\big((P_i - 1, \theta_i)_{i=1}^{p_1(f)}\big) = \eta_n^t\big((P_i - 1, \theta_i)_{i=1}^{p_2(f)}\big) + \eta_n^t\big((0, \theta_i)_{i=p_2(f)+1}^{p_1(f)}\big).$$

Recall that the second term does not depend on t. The left-hand side (as a function of n) 'turns' at $A_k^t\big((P_i - 1, \theta_i)_{i=1}^{p_1(f)}\big)$, and the first term on the right-hand side turns at

$A_k^t\big((P_i-1,\theta_i)_{i=1}^{p_2(f)}\big)$; both coordinates have a representation as in lemma 2.5 (b) (with the same σ_t). Thus when $t \gg 1$ we have

$$A_k^t\big((P_i-1,\theta_i)_{i=1}^{p_1(f)}\big) = A_k^t\big((P_i-1,\theta_i)_{i=1}^{p_2(f)}\big) \qquad \text{for} \quad k=1,\ldots,p_2(f). \tag{23}$$

(c) The value 0 at the position $a_1(T^t f)+1$ in the sequence $T^t f$ 'moves', when applying the '10'-elimination, to the position $a_1(T^t f)$ in the sequence $ET^t f$; the position of the corresponding 0 in the sequence $T^t Ef$ will be denoted by $\alpha_1(t)$. Precisely, given $f \in \Omega$ such that $Ef \neq 0$, and $t \in \mathbb{Z}$, define $\alpha_1(t) \in \mathbb{Z}$ such that the following relation is satisfied:

$$(ET^t f)(n + a_1(T^t f)) = (T^t Ef)(n+\alpha_1(t)) \qquad \text{for all} \quad n \in \mathbb{Z}.$$

We claim that $\alpha_1(t) = a_1(T^t f) - t$: for $T^t E = S^t ET^t$ implies $(T^t Ef)(n+\alpha_1(t)) = (S^t ET^t f)(n+\alpha_1(t)) = (ET^t f)(n+\alpha_1(t)+t)$, and this and the above definition imply $\alpha_1(t)+t = a_1(T^t f)$.

Again we claim that $\alpha_1(t) = a_1(T^t Ef)$ holds when $t \gg 1$: because when $t \gg 1$ in either $ET^t f$ and $T^t Ef$ the largest block is the rightmost, whose size is $\geqslant 1$ since $Ef \neq 0$; therefore from the definition of $\alpha_1(t)$ we have the assertion.

Accordingly, if $t \gg 1$ we have

$$a_1(T^t Ef) = a_1(T^t f) - t. \tag{24}$$

It follows from fact 3.1 that when $t \gg 1$ we have $a_i(ET^t f) - a_{i+1}(ET^t f) = a_i(T^t f) - a_{i+1}(T^t f) - 2$ for each $i = 1,\ldots,p_2(f)$; since $TE = SET$ the left-hand side is equal to $a_i(T^t Ef) - a_{i+1}(T^t Ef)$, so that

$$a_i(T^t Ef) - a_{i+1}(T^t Ef) = a_i(T^t f) - a_{i+1}(T^t f) - 2 \quad \text{for} \quad i=1,\ldots,p_2(f). \tag{25}$$

(d) Let $t \gg 1$ and $i = 1,\ldots,p_2(f)$. We have

$$\begin{aligned} a_i(T^t Ef) - a_{i+1}(T^t Ef) &= a_i(T^t f) - a_{i+1}(T^t f) - 2 \\ &= A_i^t\big((P_i,\theta_i)_{i=1}^{p_1(f)}\big) - A_{i+1}^t\big((P_i,\theta_i)_{i=1}^{p_1(f)}\big) - 2 \\ &= A_i^t\big((P_i-1,\theta_i)_{i=1}^{p_1(f)}\big) - A_{i+1}^t\big((P_i-1,\theta_i)_{i=1}^{p_1(f)}\big) \\ &= A_i^t\big((P_i-1,\theta_i)_{i=1}^{p_2(f)}\big) - A_{i+1}^t\big((P_i-1,\theta_i)_{i=1}^{p_2(f)}\big), \end{aligned}$$

where we have used (25), theorem 3.2, (22) and (23). On the other hand,

$$\begin{aligned} a_1(T^t Ef) = a_1(T^t f) - t &= A_1^t\big((P_i,\theta_i)_{i=1}^{p_1(f)}\big) - t \\ &= A_1^t\big((P_i-1,\theta_i)_{i=1}^{p_1(f)}\big) = A_1^t\big((P_i-1,\theta_i)_{i=1}^{p_2(f)}\big), \end{aligned}$$

where we have used (24). Hence, we find that

$$a_i(T^t Ef) = A_i^t\big((P_i-1,\theta_i)_{i=1}^{p_2(f)}\big)$$

holds, as desired. □

5. Solution to the initial-value problem for the box–ball system

Fix a state $f \in \Omega$ and write $N = p_1(f)$. In this section, we develop a method of determining a special element of $\mathfrak{S}_N^0$, (12), associated with f. As a consequence we obtain a representation of the solution to the initial-value problem (IVP) for the BBS.

Recall that $p_i = p_i(f)$ and $L_i = L_i(f)$ satisfy

$$\begin{aligned} &p_1 \geqslant \cdots \geqslant p_s > 0, \qquad s = L_1; \qquad p_i = 0 \quad \text{for} \quad i > s; \\ &L_1 \geqslant \cdots \geqslant L_N > 0, \qquad N = p_1; \qquad L_j = 0 \quad \text{for} \quad j > N. \end{aligned}$$

Choose the parameters P_i, θ_i of the soliton solution (3) as in (19). We assume (without loss of generality)

$$P_i = P_j,\ i < j \Longrightarrow \theta_i \geqslant \theta_j. \tag{26}$$

We can make the set $\mathfrak{S}_N^0 = \mathfrak{S}_N^{t=0}\left((P_i,\theta_i)_{i=1}^N\right)$ into a totally ordered set by defining $\sigma < \tau$ in $\mathfrak{S}_N^0$ if there exists i $(1 \leqslant i \leqslant N)$ such that

$$\sigma(j) = \tau(j) \qquad \text{for} \quad \forall\, j < i, \qquad \text{and} \qquad \sigma(i) < \tau(i).$$

In what follows we write σ for the smallest element under this ordering of $\mathfrak{S}_N^0$.

Define J_k for each $k = 1, \ldots, s$ (where $s = L_1$) by

$$J_k = \sigma^{-1}((p_{k+1}, p_k]) = \{j \in [p_1] \mid p_{k+1} < \sigma(j) \leqslant p_k\}. \tag{27}$$

Then it follows that (i) $P_{\sigma(j)} = k$ if $j \in J_k$, (ii) $J_k = \{j \in [p_1] \mid P_{\sigma(j)} = k\}$ and (iii) $\sum_{k=1}^s J_k = [N] = [p_1]$.

Proposition 5.1. *For each $k = 1, \ldots, s$,*

$$i, j \in J_k,\ i < j \Longrightarrow \theta_{\sigma(i)} \geqslant \theta_{\sigma(j)},$$

and hence

$$i, j \in J_k,\ i < j \Longrightarrow \sigma(i) < \sigma(j).$$

Proof. For $i, j \in J_k, i < j$,

$$\begin{aligned}\theta_{\sigma(i)} - \theta_{\sigma(j)} &= A_i^0 + \sum_{h=1}^{i-1} 2\min\{P_{\sigma(h)}, P_{\sigma(i)}\} - \theta_{\sigma(j)} \quad \text{(lemma 2.5 (b))}\\ &\geqslant -\sum_{h=1}^{i-1} 2\min\{P_{\sigma(h)}, P_{\sigma(j)}\} + \sum_{h=1}^{i-1} 2\min\{P_{\sigma(h)}, P_{\sigma(i)}\} \quad \text{(lemma 2.5 (c))}.\end{aligned}$$

Since $P_{\sigma(i)} = P_{\sigma(j)} = k$ the right-hand side is 0, which is the first assertion.

The proof of the second assertion goes as follows:

If $\theta_{\sigma(i)} > \theta_{\sigma(j)}$ then the assumption (26) implies $\sigma(i) < \sigma(j)$.

If $\theta_{\sigma(i)} = \theta_{\sigma(j)}$ then the definition of σ as the smallest element of $\mathfrak{S}_{p_1}^0$ implies $\sigma(i) < \sigma(j)$: for, if not so, that is, if $\sigma(i) > \sigma(j)$ holds, then, defining $\tilde{\sigma}$ as $\tilde{\sigma}(i) = \sigma(j), \tilde{\sigma}(j) = \sigma(i)$ and $\tilde{\sigma}(h) = \sigma(h)$ for the other h, we have $\tilde{\sigma}(h) = \sigma(h)$ for $h < i$, and $\tilde{\sigma}(i) = \sigma(j) < \sigma(i)$, so that $\tilde{\sigma} < \sigma$, which contradicts the assumption on σ. □

The J_k's therefore determine σ completely. Let us describe a method for finding the sets J_k's.

First, we introduce the concept of a multiset (or a set allowing repeated elements) consisting of coordinates of turning points of a piecewise linear function.

For a positive integer N let $\mathcal{P}_N$ denote the set of all real-valued convex piecewise linear function on $\mathbb{R}$ such that: (i) the slope on each interval on which the function is linear is a negative integer or 0; (ii) the slope on the left half infinite interval is $-N$; and (iii) the slope on the right half infinite interval is 0. For instance, the function $n \mapsto \eta_n^t\left((P_i,\theta_i)_{i=1}^N\right)$, (3), for non-negative P_i's (and for any t) belongs to $\mathcal{P}_N$.

We define an N-element multiset on $\mathbb{R}$ to be a mapping $\nu : \mathbb{R} \to \mathbb{Z}_{\geqslant 0}$ such that the set $\operatorname{supp}\nu = \nu^{-1}(\mathbb{Z}_{>0})$ is finite and $\sum_{n\in\operatorname{supp}\nu} \nu(n) = N$. The set of all such multisets is denoted by $\left(\!\binom{\mathbb{R}}{N}\!\right)$. If $\operatorname{supp}\nu = \{x_1, \ldots, x_m\}$ and $\nu(x_i) = \lambda_i$, then we write $\{\underbrace{x_1, \ldots, x_1}_{\lambda_1}, \ldots, \underbrace{x_m, \ldots, x_m}_{\lambda_m}\}$ or $\left\{x_1^{\lambda_1}, \ldots, x_m^{\lambda_m}\right\}$ for the multiset.

Let $\eta \in \mathcal{P}_N$; if the set of coordinates of all turning points of η is $\{x_1, \ldots, x_m\}$ and the slopes increase by λ_i when passing through x_i, then we put $\mathfrak{d}_N(\eta) = \{x_1^{\lambda_1}, \ldots, x_m^{\lambda_m}\} \in \left(\!\binom{\mathbb{R}}{N}\!\right)$. This defines a one-to-one correspondence $\mathfrak{d}_N : \mathcal{P}_N \to \left(\!\binom{\mathbb{R}}{N}\!\right)$. From propositions 2.1 and 2.2 (cf figure 1), the image of $\eta_n^0\big((P_i, \theta_i)_{i=1}^N\big)$ by this mapping is a multiset consisting of A_k^0, $k = 1, \ldots, N$, given in (7).

Put $\mathcal{P} = \bigcup_{N\geqslant 0} \mathcal{P}_N$ and $\mathcal{D} = \bigcup_{N\geqslant 0} \left(\!\binom{\mathbb{R}}{N}\!\right)$. In either set, the sum of two elements can be defined naturally as the sum of functions (mappings). In $\mathcal{P}$, if $\eta \in \mathcal{P}_N$ and $\eta' \in \mathcal{P}_{N'}$ then $\eta + \eta' \in \mathcal{P}_{N+N'}$. Define $\mathfrak{d} : \mathcal{P} \to \mathcal{D}$ to be $\mathfrak{d}(\eta) = \mathfrak{d}_N(\eta)$ for $\eta \in \mathcal{P}_N$. Then $\mathfrak{d}(\eta + \eta') = \mathfrak{d}(\eta) + \mathfrak{d}(\eta')$.

It follows from proposition 2.1 (c), (d) that

$$\eta_n^0\big((P_i - 1, \theta_i)_{i=1}^{p_1}\big) = \eta_n^0\big((P_i - 1, \theta_i)_{i=1}^{p_2}\big) + \eta_n^0\big((0, \theta_i)_{i=p_2+1}^{p_1}\big).$$

(If $p_2 = p_1$ then the second term on the right-hand side does not exist; if $p_2 = 0$ then the first term does not exist. In what follows we will not mention such exceptional cases.) The function on the left turns at $A_i^0\big((P_i - 1, \theta_i)_{i=1}^{p_1}\big) = A_i^0\big((P_i, \theta_i)_{i=1}^{p_1}\big) + 2(i-1) = a_i(f) + 2(i-1)$ by lemma 2.5 (b) and corollary 3.3. The function given by the first term on the right turns at $a_i(Ef)$ by theorem 4.2. Thus, mapping the above equation by $\mathfrak{d}$ yields

$$\{a_i(f) + 2(i-1)\}_{i=1}^{p_1} = \{a_i(Ef)\}_{i=1}^{p_2} \cup \{\theta_i\}_{i=p_2+1}^{p_1}, \tag{28}$$

where the right-hand side is the sum of multisets.

Whether $i \in [p_1]$ belongs to J_1 or not can be judged as follows: write $\bar{a}_i^{(1)} = a_i(f) + 2(i-1)$ and $a_i^{(1)} = a_i(Ef)$. Let $i \in [p_1]$; from (28) we have

(i) if $\bar{a}_i^{(1)}$ is not equal to any $a_h^{(1)}$, $h \in [p_2]$, then $i \in J_1$ (clear);
(ii) if $\bar{a}_i^{(1)}$ is equal to $a_h^{(1)}$ for some $h \in [p_2]$, and $\bar{a}_j^{(1)} \neq a_h^{(1)}$ for $j \neq i$, then $i \notin J_1$ (again, clear); or
(iii) if $\bar{a}_i^{(1)}$ is equal to $a_h^{(1)}$ for some $h \in [p_2]$, and $\bar{a}_j^{(1)} = a_h^{(1)}$ for some j other than i, then (iiia) $i \notin J_1$ if $i = 1$ or if $i > 1$ and $\bar{a}_{i-1}^{(1)} > \bar{a}_i^{(1)}$ or (iiib) $i \in J_1$ otherwise (σ being the smallest element in $\mathfrak{S}_{p_1}^0$).

Summarizing, if we write $I_0 = \{1, \ldots, p_1\}$ and

$$I_1 = \big\{i_1^{(1)}, \ldots, i_{p_2}^{(1)}\big\}, \qquad i_h^{(1)} = \min\big\{i \in I_0 \,\big|\, \bar{a}_i^{(1)} = a_h^{(1)}\big\}$$

then $J_1 = I_0 \backslash I_1$.

Similarly, from

$$\big\{\bar{a}_i^{(2)}\big\}_{i\in I_1} = \big\{a_i^{(2)}\big\}_{i=1}^{p_3} \cup \big\{\theta_i\big\}_{i=p_3+1}^{p_2}, \qquad \bar{a}_{i_h^{(1)}}^{(2)} = a_h^{(1)} + 2(h-1), \qquad a_i^{(2)} = a_i(E^2 f),$$

we obtain $J_2 = I_1 \backslash I_2$, where

$$I_2 = \big\{i_1^{(2)}, \ldots, i_{p_3}^{(2)}\big\}, \qquad i_h^{(2)} = \min\big\{i \in I_1 \,\big|\, \bar{a}_i^{(2)} = a_h^{(2)}\big\}.$$

Continuing this procedure we obtain all J_k's. Thus we have established a method for determining J_k's and, therefore, σ, which we summarize in the following proposition.

Proposition 5.2. *Let* $I_0 = \{1, \ldots, p_1\}$. *Write*

$$a_i^{(k)} = a_i(E^k f) \qquad (i = 1, \ldots, p_{k+1}; k = 0, 1, \ldots, s-1)$$

so that $a_i^{(0)} = a_i$.

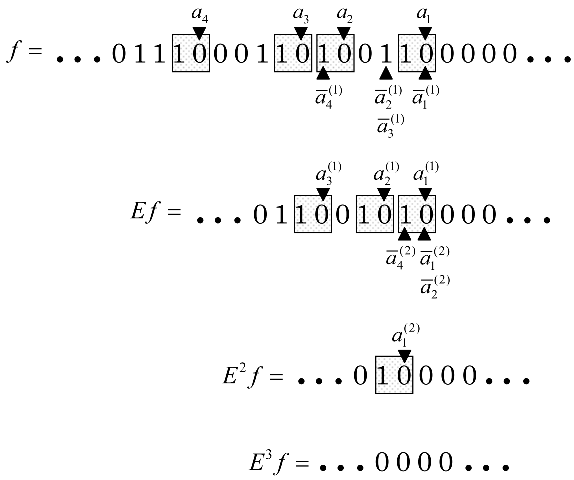

Figure 7. Schematic example of the procedure to find J_k's according to proposition 5.2. See also example 5.3. Positions having coordinates a_i's and others are indicated. In each state (sequence) the '10's surrounded by dotted squares will be eliminated when E is applied.

(a) *For* $k = 1, \ldots, s-1$, *we define* $\bar{a}_i^{(k)}, i_h^{(k)}$ *and* I_k *by the recurrence*

$$\bar{a}^{(k)}_{i_j^{(k-1)}} = a_j^{(k-1)} + 2(j-1) \qquad (j = 1, \ldots, p_k),$$
$$i_h^{(k)} = \min\left\{i \in I_{k-1} \,\middle|\, \bar{a}_i^{(k)} = a_h^{(k)}\right\} \qquad (h = 1, \ldots, p_{k+1}),$$
$$I_k = \left\{i_h^{(k)} \,\middle|\, h = 1, \ldots, p_{k+1}\right\}.$$

Then the following equality between multisets:

$$\left\{\bar{a}_i^{(k)}\right\}_{i \in I_{k-1}} = \left\{a_i^{(k)}\right\}_{i=1}^{p_{k+1}} \cup \left\{\theta_i\right\}_{i=p_{k+1}+1}^{p_k}$$

holds, and hence

$$J_k = I_{k-1} \backslash I_k$$

follows for each $k = 1, \ldots, s$.

(b) *For each* k *let the elements of* J_k *be* $j_{p_{k+1}+1}, \ldots, j_{p_k}$ *where* $j_{p_{k+1}+1} < \cdots < j_{p_k}$. *Then*

$$\sigma(j_h) = h \qquad (h = 1, \ldots, p_1).$$

We note that part (b) is a consequence of the second assertion of proposition 5.1.

Example 5.3. We demonstrate the procedure for finding I_k and J_k. See also figure 7. Suppose

$$f = \cdots 0\,1\,1\,1\,0\,0\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,0 \cdots \in \Omega.$$

Step 0. From f we read $p_1 = 4$ and where the positions of a_1, a_2, a_3, a_4 are $I_0 = [p_1] = [4] = \{1, 2, 3, 4\}$.

Step 1. Find $\bar{a}_h^{(1)} = a_h + 2(h-1)$ $(h \in [p_1] = [4])$. From Ef then $p_2 = 3$ and the positions of $a_1^{(1)}, a_2^{(1)}, a_3^{(1)}$ are read off. Thus $i_h^{(1)}$ $(h \in [p_2] = [3])$ are

$$i_1^{(1)} = \min\left\{i \in I_0 \,\middle|\, \bar{a}_i^{(1)} = a_1^{(1)}\right\} = \min\{1\} = 1,$$
$$i_2^{(1)} = \min\left\{i \in I_0 \,\middle|\, \bar{a}_i^{(1)} = a_2^{(1)}\right\} = \min\{2, 3\} = 2,$$
$$i_3^{(1)} = \min\left\{i \in I_0 \,\middle|\, \bar{a}_i^{(1)} = a_3^{(1)}\right\} = \min\{4\} = 4.$$

Hence

$$I_1 = \left\{ i_h^{(1)} \,\middle|\, h \in [p_2] = [3] \right\} = \{1, 2, 4\}, \qquad J_1 = I_0 \backslash I_1 = \{3\}.$$

Step 2. Find $\bar{a}^{(2)}_{i_h^{(1)}} = a_h^{(1)} + 2(h-1)$ ($h \in [p_2] = [3]$), that is,

$$\bar{a}_1^{(2)} = a_1^{(1)}, \qquad \bar{a}_2^{(2)} = a_2^{(1)} + 2, \qquad \bar{a}_4^{(2)} = a_3^{(1)} + 4.$$

From $E^2 f$ we find $p_3 = 1$ and positions of $a_1^{(2)}$. Thus $i_h^{(2)}$ ($h \in [p_3] = [1]$) is

$$i_1^{(2)} = \min\left\{ i \in I_1 \,\middle|\, \bar{a}_i^{(2)} = a_1^{(2)} \right\} = \min\{1\} = 4.$$

Hence

$$I_2 = \left\{ i_h^{(2)} \,\middle|\, h \in [p_3] = [1] \right\} = \{4\}, \qquad J_2 = I_1 \backslash I_2 = \{1, 2\}.$$

Step 3. $E^3 f$ is zero. Therefore $s = 3$ (recall $s = L_1$) and $p_4 = 0$. Hence

$$I_3 = \emptyset, \qquad J_3 = I_2 \backslash I_3 = I_2 = \{4\}.$$

We have obtained all J_k's.

The permutation σ is determined as follows: we write the elements of J_3, J_2 and J_1 in a row in this order, where the elements of each J_k are to be arranged in ascending order, and read them as j_1, j_2, j_3, j_4,

$$\begin{array}{cccc} \overbrace{4}^{J_3} & \overbrace{1 \quad 2}^{J_2} & & \overbrace{3}^{J_1} \\ \shortparallel & \shortparallel \quad \shortparallel & & \shortparallel \\ j_1 & j_2 \quad j_3 & & j_4. \end{array}$$

Then $\sigma(j_h) = h$, that is,

$$\begin{array}{cccc} \sigma(4) & \sigma(1) & \sigma(2) & \sigma(3) \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ 1 & 2 & 3 & 4. \end{array}$$

As a corollary of the above propositions, we obtain a representation of the solution to the IVP for the BBS.

Theorem 5.4 (solution to the IVP for the BBS)**.** *Let $f \in \Omega$. The state of the BBS at t corresponding to the initial state f is given by*

$$u_n^t = (T^t f)(n) = \eta_{n+1}^{t-1} - \eta_{n+1}^t - \eta_n^{t-1} + \eta_n^t,$$

where

$$\eta_n^t = \max\left\{ 0, \max_{\substack{J \subset [p_1] \\ J \neq \emptyset}} \left[\sum_{i \in J} \left(a_i + \sum_{j=1}^{i-1} 2\min\{W_i, W_j\} + tW_i - n \right) - \sum_{\substack{i,j \in J \\ i \neq j}} \min\{W_i, W_j\} \right] \right\}, \tag{29}$$

$a_i = a_i(f)$ and $W_i = P_{\sigma(i)}$. (We call W_i the amplitude of the ith *soliton.) The values of the W_i's are obtained as follows: let $i \in [p_1]$; find a number $k \in [s]$ such that $i \in J_k$ (recall $s = L_1$); then $W_i = k$.*

Proof. Suppose we have chosen the values of parameters P_i, θ_i as in theorem 3.2. The function η_n^t, (3), can be written as

$$\eta_n^t = \max\left\{0, \max_{\substack{J\subset[N]\\J\neq\emptyset}}\left[\sum_{i\in J}(\theta_{\sigma(i)} + tP_{\sigma(i)} - n) - \sum_{\substack{i,j\in J\\i\neq j}}\min\{P_{\sigma(i)}, P_{\sigma(j)}\}\right]\right\}.$$

Putting $t = 0$ in (21) in corollary 3.3

$$\theta_{\sigma(i)} = a_i + \sum_{j=1}^{i-1} 2\min\{P_{\sigma(i)}, P_{\sigma(j)}\}.$$

Substituting this into the above equation yields (29). The method of finding W_i follows from proposition 5.1. □

Example 5.5. Suppose

$$f = \cdots 0111000110100011000\cdots$$

as in example 5.3. Then $p_1 = 4$. We have already known J_k's,

$$J_1 = \{3\}, \qquad J_2 = \{1, 2\}, \qquad J_3 = \{4\}.$$

Checking whether $i \in J_k$ for $i = 1, 2, 3, 4$,

$1 \in J_2$, therefore $W_1 = 2$;
$2 \in J_2$, therefore $W_2 = 2$;
$3 \in J_1$, therefore $W_3 = 1$;
$4 \in J_3$, therefore $W_4 = 3$.

Remark 5.6. It follows that the values of W_i's can also be obtained from the following algorithm.

`Input`: $f \in \Omega$.
`Output`: a finite sequence of sequences of 1, 0 and ‘ ’ (SPACE)
`Begin`
  Set $f^{(0)} \leftarrow f$ and $k \leftarrow 0$.
  `While` $f^{(k)} \neq 0$ do
1. $g \leftarrow f^{(k)}$.
2. In each block of $1'$s in g, replace the leftmost 1 by SPACE. In each block of $0'$s of finite length, replace the rightmost 0 by SPACE. Update g.
3. In g, there can be the blocks of $1'$s whose right is SPACE. Translate each such block, in order from the rightmost block to the left one, to the right up to the position where its right is 1 or 0. Update g.
4. In g, there can be the blocks of $0'$s whose left is SPACE. Translate each such block, in order from the leftmost block to the right one, to the left up to the position where its left is 1 or 0. Update g.
5. Set $f^{(k+1)} \leftarrow g$ and $k \leftarrow k + 1$.

`End`

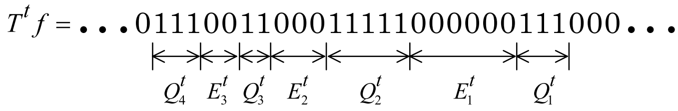

Figure 8. Variables of the ultradiscrete Toda molecule equation.

Write the sequences $f^{(k)}, k = 0, 1, \ldots,$ in order from up to down. We have a two-dimensional array of 1, 0 and SPACE, in which there are clusters of 1's. Then the depth of the ith cluster is W_i.

For example, suppose $f = \ldots 011100011010011000 \ldots$ is given. Then the algorithm goes as follows:

$$\begin{aligned}
f^{(0)} &= \ldots\texttt{011100011010011000}\ldots \\
g &= \ldots\texttt{0 1100 1 0 1000}\ldots \quad \text{(applied 1 and 2)} \\
g &= \ldots\texttt{0 1100 10 1000}\ldots \quad \text{(applied 3)} \\
f^{(1)} &= \ldots\texttt{0 1100 10 1000}\ldots \quad \text{(applied 4 and 5)} \\
g &= \ldots\texttt{0 10 000}\ldots \quad \text{(applied 1 and 2)} \\
g &= \ldots\texttt{0 10 000}\ldots \quad \text{(applied 3)} \\
f^{(2)} &= \ldots\texttt{0 10000}\ldots \quad \text{(applied 4 and 5)} \\
g &= \ldots\texttt{0 0000}\ldots \quad \text{(applied 1 and 2)} \\
g &= \ldots\texttt{0 0000}\ldots \quad \text{(applied 3)} \\
f^{(3)} &= \ldots\texttt{00000}\ldots \quad \text{(applied 4 and 5).}
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
f^{(0)} &= \ldots\texttt{011100011010011000}\ldots \\
f^{(1)} &= \ldots\texttt{0 1100 10 1000}\ldots \\
f^{(2)} &= \ldots\texttt{0 10000}\ldots \\
f^{(3)} &= \ldots\texttt{00000}\ldots.
\end{aligned}$$

There are four clusters in this array. The depth of each is (from right to left) 2, 2, 1 and 3, respectively, and hence $W_1 = 2$, $W_2 = 2$, $W_3 = 1$ and $W_4 = 3$.

5.1. Solution to the initial-value problem for the ultradiscrete Toda molecule equation

At this point, we remark that the above results enable us to write an expression for the solution to the IVP of the ultradiscrete version of the Toda molecule equation.

Let $f \in \Omega$, and consider the state $T^t f$ of the BBS at t. Let Q_m^t $(m = 1, \ldots, N)$ denote the length of the mth block of 1's, and let E_m^t $(m = 1, \ldots, N-1)$ denote the length of the mth block of 0's. (Blocks are numbered from right to left; see figure 8.) They satisfy a set of equations

$$\begin{aligned}
Q_m^{t+1} &= \min\left\{ \sum_{l=m}^{N} Q_l^t - \sum_{l=m+1}^{N} Q_l^{t+1}, E_{m-1}^t \right\} && (m = 1, \ldots, N), \\
E_m^{t+1} &= Q_m^t + E_m^t - Q_{m+1}^{t+1} && (m = 1, \ldots, N-1),
\end{aligned} \tag{30}$$

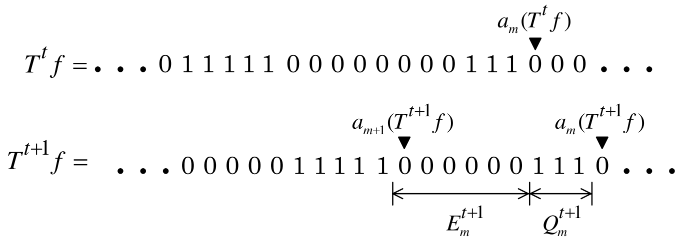

Figure 9. The variables are expressed in terms of the 'position of block' variables a_m's.

where $E_0^t = +\infty$. These are obtained by ultradiscretizing

$$I_m^{t+1} = \frac{\prod_{j=1}^{m} I_j^t}{\prod_{j=1}^{m-1} I_j^{t+1}} + V_m^t, \qquad V_m^{t+1} = \frac{I_{m+1}^t V_m^t}{I_m^{t+1}}$$

or, equivalently,

$$I_m^{t+1} = I_m^t + V_m^t - V_{m-1}^{t+1}, \qquad V_m^{t+1} = \frac{I_{m+1}^t V_m^t}{I_m^{t+1}}$$

through

$$-\epsilon \log I_{N-m+1}^t \longrightarrow Q_m^t, \qquad -\epsilon \log V_{N-m}^t \longrightarrow E_m^t$$

as $\epsilon \to 0$. The set of the above equations is known as the Toda molecule equation (see [4] and references therein), and hence, we call (30) the ultradiscrete Toda molecule equation.

It is easily seen that the new variables can be expressed in terms of a_m's (see figure 9),

$$Q_m^{t+1} = a_m(T^{t+1} f) - a_m(T^t f), \qquad E_m^{t+1} = a_m(T^t f) - a_{m+1}(T^{t+1} f). \tag{31}$$

Since $a_m(T^t f) = A_m^t = \Psi_m^t - \Psi_{m-1}^t$ we have

$$Q_m^{t+1} = \Psi_m^{t+1} - \Psi_{m-1}^{t+1} - \Psi_m^t + \Psi_{m-1}^t, \qquad E_m^{t+1} = \Psi_m^t - \Psi_{m-1}^t - \Psi_{m+1}^{t+1} + \Psi_m^{t+1}. \tag{32}$$

The same argument as in the proof of theorem 5.4 leads to an expression for Ψ's in terms of a_i's and W_i's,

$$\begin{aligned}
\Psi_m^t &= \max_{\substack{J\subset[N]\\ |J|=m}} \left[\sum_{i\in J} (\theta_i + tP_i) - \sum_{\substack{i,j\in J\\ i\neq j}} \min\{P_i, P_j\} \right] \\
&= \max_{\substack{J\subset[N]\\ |J|=m}} \left[\sum_{i\in J} (\theta_{\sigma(i)} + tP_{\sigma(i)}) - \sum_{\substack{i,j\in J\\ i\neq j}} \min\{P_{\sigma(i)}, P_{\sigma(j)}\} \right] \\
&= \max_{\substack{J\subset[N]\\ |J|=m}} \left[\sum_{i\in J} \left(a_i + \sum_{j=1}^{i-1} 2\min\{W_i, W_j\} + tW_i \right) - \sum_{\substack{i,j\in J\\ i\neq j}} \min\{W_i, W_j\} \right].
\end{aligned} \tag{33}$$

Equation (32) together with (33) gives the solution to the IVP for the ultradiscrete Toda molecule equation (30).

6. Solution to the initial-value problem for the periodic box–ball system

Let $L \geqslant 3$ and let $\Omega_L = \{f \mid f : [L] \to \{0, 1\} \text{ such that } \# f^{-1}(\{1\}) < L/2\}$. Define $T_L : \Omega_L \to \Omega_L$ in the same manner as T in figure 3. The pair (Ω_L, T_L) is called a periodic box–ball system, or PBBS for short, of length L [9, 16]. An element of Ω_L is called a state, and the mapping T_L the time evolution.

Theorem 6.1 (solution to the IVP for the PBBS)**.** *Let f be a state of the PBBS: $f \in \Omega_L$. Shifting the origin if necessary, we can assume that the position $n = 1$ in f is not 'marked' (see the caption of figure 3 for the definition of the term 'marked'). Let the number of blocks in f be N. Thinking of $n = L$ as the rightmost position and $n = 1$ the leftmost, we define the position of* ith *block a_i and the amplitude W_i $(i = 1, \ldots, N)$ in the same manner as for the BBS (cf theorem 5.4). Then the state $T_L^t f$ of the PBBS at t corresponding to the initial state f is given by*

$$\begin{aligned} \left(T_L^t f\right)(n) &= \eta_{n+1}^{t-1} - \eta_{n+1}^{t} - \eta_n^{t-1} + \eta_n^t; \\ \eta_n^t &= \max_{\substack{m_i \in \mathbb{Z} \\ (i=1,\ldots,N)}} \left[\sum_{i=1}^{N} m_i \left(b_i + t W_i - n\right) - \sum_{i=1}^{N} \sum_{j=1}^{N} m_i \Xi_{ij} m_j \right], \end{aligned} \tag{34}$$

where

$$b_i = a_i + \sum_{j=1}^{i-1} 2 \min\{W_i, W_j\} + W_i + \frac{Z_i}{2}, \tag{35}$$

$$\Xi_{ij} = \frac{Z_i}{2} \delta_{ij} + \min\{W_i, W_j\}, \tag{36}$$

$$Z_i = L - \sum_{j=1}^{N} 2 \min\{W_i, W_j\} \tag{37}$$

with δ_{ij} being Kronecker's delta.

Proof. Let $\widetilde{\Omega}_L = \{\tilde{f} \mid \tilde{f} : \mathbb{Z} \to \{0, 1\} \text{ such that } \tilde{f}(n + L) = \tilde{f}(n) \text{ for all } n \in \mathbb{Z}, \text{ and that } \#(\tilde{f}^{-1}(\{1\}) \cap [L]) < L/2\}$. For every $f \in \Omega_L$ there exists $\tilde{f} \in \widetilde{\Omega}_L$ such that $\tilde{f}|_{[L]} = f$. Define $\widetilde{T}_L : \widetilde{\Omega}_L \to \widetilde{\Omega}_L$ as $\widetilde{T}_L(\tilde{f}) = T_L(\tilde{f}|_{[L]})$. The correspondence between the dynamical system $(\widetilde{\Omega}_L, \widetilde{T}_L)$ and the PBBS (Ω_L, T_L) is one to one.

Let f be a state of the PBBS: $f \in \Omega_L$. For each non-negative integer S, define a state of the BBS $f_S \in \Omega$ to be

$$f_S(j) = \begin{cases} f(n) & \text{if} \quad -SL + 1 \leqslant j \leqslant (S+1)L \quad \text{and} \quad j \equiv n \bmod L \\ 0 & \text{if} \quad j \leqslant -SL \text{ or } (S+1)L + 1 \leqslant j. \end{cases}$$

Then we observe that the limit $\lim_{S\to\infty} T^t f_S$ should be in $\widetilde{\Omega}_L$, and its restriction to $[L]$ is the state $T_L^t f$ at t of the PBBS corresponding to the initial state f.

It follows from theorem 5.4 that

$$(T^t f_S)(n) = \tilde{\eta}^{t-1}_{S,n+1} - \tilde{\eta}^{t}_{S,n+1} - \tilde{\eta}^{t-1}_{S,n} + \tilde{\eta}^{t}_{S,n};$$

$$\tilde{\eta}^t_{S,n} = \max\left\{0, \max_{\substack{J\subset[(2S+1)N]\\ J\neq\emptyset}}\left[\sum_{i\in J}\left(\tilde{a}_i + \sum_{j=1}^{i-1} 2\min\{\tilde{W}_i, \tilde{W}_j\} + t\tilde{W}_i - n\right) - \sum_{\substack{i,j\in J\\ i\neq j}} \min\{\tilde{W}_i, \tilde{W}_j\}\right]\right\}, \tag{38}$$

where $\tilde{a}_i$ and $\tilde{W}_i$ are the position and the amplitude, respectively, of the ith block of f_S. Clearly, we have

$$\tilde{a}_{i+N} = \tilde{a}_i - L, \qquad \tilde{W}_{i+N} = \tilde{W}_i$$

for $i \in [2SN]$, and

$$a_i = \tilde{a}_{i+SN} = \tilde{a}_i - SL,$$
$$W_i = \tilde{W}_k \text{ for all } k \in [(2S+1)N] \text{ such that } k \equiv i \bmod N$$

for $i \in [N]$.

If we write $\tilde{\theta}_i = \tilde{a}_i + \sum_{j=1}^{i-1} 2\min\{\tilde{W}_i, \tilde{W}_j\}$ for $i \in [(2S+1)N]$ then

$$\tilde{\theta}_{i+N} - \tilde{\theta}_i = -L + \sum_{j=1}^{N} 2\min\{\tilde{W}_i, W_j\} < -2\# f^{-1}(\{1\}) + \sum_{j=1}^{N} 2\min\{\tilde{W}_i, W_j\} \leqslant 0$$

('$<$' follows from the assumption on the number of 1's in f; and '$\leqslant$' from $\# f^{-1}(\{1\}) = \sum_{j=1}^N W_j \geqslant \sum_{j=1}^N \min\{\tilde{W}_i, W_j\}$) and therefore $\tilde{\theta}_i > \tilde{\theta}_{i+N}$.

Put $T_k = \{k, N+k, 2N+k, \dots, 2SN+k\}$ for $k = 1, \dots, N$, so that $\bigcup T_k = [(2S+1)N]$. Let $\mathfrak{T}$ denote the set of all subsets of $[(2S+1)N]$, and let $\mathfrak{T}_{n_1,\dots,n_N} = \{J \in \mathfrak{T} \mid \#(J \cap T_k) = n_k (k = 1, \dots, N)\}$ for each $(n_1, \dots, n_N)$. Then $\mathfrak{T}$ is the disjoint union of $\mathfrak{T}_{n_1,\dots,n_N}$ for all possible $(n_1, \dots, n_N)$, hence

$$\tilde{\eta}^t_{S,n} = \max_{J\in\mathfrak{T}}[\cdots] = \max_{\substack{0\leqslant n_j\leqslant 2S+1\\ (j=1,\dots,N)}}\left[\max_{J\in\mathfrak{T}_{n_1,\dots,n_N}}[\cdots]\right].$$

Since $\tilde{\theta}_i > \tilde{\theta}_{i+N}$, the inner bracket $[\cdots]$ on the right attains its maximum at

$$J_{n_1,\dots,n_N} = \{1, N+1, 2N+1, \dots, (n_1-1)N+1\}$$
$$\bigcup\{2, N+2, 2N+2, \dots, (n_2-1)N+2\}\bigcup\cdots\bigcup\{N, 2N, 3N, \dots, n_N N\}.$$

Since $\tilde{\theta}_{i+mN} = \tilde{\theta}_i - mZ_i$ with Z_i defined in (37)

$$\sum_{i\in J_{n_1,\dots,n_N}} \tilde{\theta}_i = \sum_{i=1}^{N}\sum_{m=0}^{n_i-1} \tilde{\theta}_{i+mN} = \sum_{i=1}^{N}\sum_{m=0}^{n_i-1}(\tilde{\theta}_i - mZ_i) = \sum_{i=1}^{N}\left(n_i\tilde{\theta}_i - \frac{n_i(n_i-1)}{2}Z_i\right),$$

$$\sum_{\substack{i,j\in J_{n_1,\dots,n_N}\\ i\neq j}} \min\{\tilde{W}_i, \tilde{W}_j\} = \sum_{i=1}^{N}\sum_{j=1}^{N} n_i n_j \min\{W_i, W_j\} - \sum_{i=1}^{N} n_i W_i.$$

Thus we have

$$\tilde{\eta}^t_{S,n} = \max_{\substack{0\leqslant n_j\leqslant 2S+1\\ (j=1,\dots,N)}}\left[\sum_{i=1}^{N}\left(n_i\left(\tilde{\theta}_i + W_i + \frac{Z_i}{2} + tW_i - n\right) - \frac{n_i^2}{2}Z_i\right) - \sum_{i=1}^{N}\sum_{j=1}^{N} n_i n_j \min\{W_i, W_j\}\right].$$

Define b_i, Ξ_{ij} as in (35), (36), and substitute $n_i = m_i + S$ in $\tilde{\eta}$. Then, using $\sum_{j=1}^{N} \Xi_{ij} = L/2$ and $\sum_{i=1}^{N} W_i = \# f^{-1}(\{1\})$, we have

$$\tilde{\eta}_{S,n}^{t} = \eta_{S,n}^{t} + C_S + (\# f^{-1}(\{1\})) \cdot St - NSn,$$

where

$$\begin{aligned} \eta_{S,n}^{t} &= \max_{\substack{-S \leqslant m_j \leqslant S+1 \\ (j=1,\ldots,N)}} \left[\sum_{i=1}^{N} m_i (b_i + tW_i - n) - \sum_{i=1}^{N} \sum_{j=1}^{N} m_i \Xi_{ij} m_j \right], \\ C_S &= -\frac{S^2 NL}{2} + S \sum_{i=1}^{N} \left(\tilde{\theta}_i + W_i + \frac{Z_i}{2} \right). \end{aligned} \tag{39}$$

Hence it follows from (38) that

$$(T^t f_S)(n) = \eta_{S,n+1}^{t-1} - \eta_{S,n+1}^{t} - \eta_{S,n}^{t-1} + \eta_{S,n}^{t},$$

where the other terms on the right have been cancelled out.

Now we want to take the limit $S \to \infty$. In each term on the right-hand side the parameter S enters only in the range of the variable m_j. So, taking the limit in (39), we have

$$\lim_{S \to \infty} \eta_{S,n}^{t} = \max_{\substack{m_j \in \mathbb{Z} \\ (j=1,\ldots,N)}} \left[\sum_{i=1}^{N} m_i (b_i + tW_i - n) - \sum_{i=1}^{N} \sum_{j=1}^{N} m_i \Xi_{ij} m_j \right].$$

If the matrix $(\Xi_{ij})_{1 \leqslant i,j \leqslant N}$ were positive definite, then the maximum in the above equation exists, and hence, $\lim \eta_{S,n}^{t}$ exists, and we have the assertion of the theorem.

To show that $(\Xi_{ij})_{1 \leqslant i,j \leqslant N}$ is positive definite it is sufficient to show $I(x) = \sum \sum x_i \Xi_{ij} x_j > 0$ for nonzero $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. Recall $Z_i, W_i > 0$. Suppose that $W_{i_1} \leqslant \cdots \leqslant W_{i_N}$. Accordingly, we write $U_k = W_{i_k}$ and $y_k = x_{i_k}$ $(k = 1, \ldots, N)$. Then, for any $x \neq 0$,

$$\begin{aligned} I(x) &= \sum_{i=1}^{N} \frac{Z_i}{2} x_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \min\{U_i, U_j\} y_i y_j \\ &= \sum_{i=1}^{N} \frac{Z_i}{2} x_i^2 + \sum_{i=1}^{N} U_i y_i^2 + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} 2U_i y_i y_j \\ &= \sum_{i=1}^{N} \frac{Z_i}{2} x_i^2 + \sum_{i=1}^{N} U_i y_i^2 + \sum_{i=1}^{N-1} 2U_i y_i (y_{i+1} + \cdots + y_N) \\ &= \sum_{i=1}^{N} \frac{Z_i}{2} x_i^2 + \sum_{i=1}^{N-1} U_i [(y_i + \cdots + y_N)^2 - (y_{i+1} + \cdots + y_N)^2] + U_N y_N^2 \\ &= \sum_{i=1}^{N} \frac{Z_i}{2} x_i^2 + U_1 (y_1 + \cdots + y_N)^2 + (U_2 - U_1)(y_2 + \cdots + y_N)^2 + \cdots + (U_N - U_{N-1}) y_N^2 \\ &> 0. \end{aligned}$$

This completes the proof. □

We note that the right-hand side of (34) is an ultradiscretization of a theta function [22, 20]. The authors previously presented an algorithm for obtaining the solution to the IVP for the PBBS [17]. The above-obtained formula seems more explicit than the previous one. We also note that the same problem was also studied in [23] by a different approach; compared with theirs, our method for obtaining the values of parameters is fairly easy, and gives the definite meanings to the parameters.

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